Variants of Schanuel's conjecture

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Introduction, January 2018

This is a collection of variants of Schanuel's conjecture and the known dependencies between them. It was originally written in 2007, and made available for a time on my webpage. I have been asked by a few people to make it available again and have taken the opportunity to make some minor revisions now in January 2018. The treatment is far from exhaustive of the literature, and for the most part consists of statements which were under discussion in the Oxford logic group between around 2002 and 2007. Many of these appeared in Boris Zilber's papers [Zil00], [Zil02], and [Zil03]. The general idea (although not much stressed in this article) is that Schanuel-type statements are obtained by counting the degrees of freedom in a system of equations, and that if this is negative then the system has no solutions.

Some parts of this article are now obsolete. The section on generic powers has been superseded by the paper [BKW10]. Conjecture 3.3 is false but the issue of what part of Schanuel's conjecture is first-order expressible is discussed in [KZ14]. The relationship between non-standard integer powers and the CIT is explained in chapter 6 of [Bay09].

Conventions

We adopt the convention that theorems are unconditionally proved statements (about numbers, fields, exponential maps, etc,) and propositions are (unconditionally proved) relationships between conjectures. Most proofs are omitted. Algebraic varieties defined over a subfield of the complex numbers $\mathbb{C}$ are identified with their $\mathbb{C}$-points. Throughout the article we write $\operatorname{td}_{\mathbb{Q}}(x_1, \ldots, x_n)$ to mean the transcendence degree of the field extension $\mathbb{Q}(x_1, \ldots, x_n)/\mathbb{Q}$, and $\operatorname{ldim}_{\mathbb{Q}}(x_1, \ldots, x_n)$ to mean the $\mathbb{Q}$-linear dimension of the $\mathbb{Q}$-vector space spanned by $x_1, \ldots, x_n$.

Contents

1 Statements of Schanuel's conjecture

We begin with Stephen Schanuel's original conjecture, in several equivalent formulations.

Conjecture 1.1 (Schanuel's conjecture (SC))**.** *Let $a_1, \ldots, a_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then* $\mathrm{td}_{\mathbb{Q}}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) \geqslant n$.

Conjecture 1.2 ((SC), version 2)**.** *Let $a_1, \ldots, a_n$ be complex numbers and suppose that* $\mathrm{td}_{\mathbb{Q}}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) < n$. *Then there are integers $m_1, \ldots, m_n$, not all zero, such that* $\sum_{i=1}^{n} m_i a_i = 0$.

We define a "predimension function" δ on n-tuples of complex numbers by

$$\delta(a_1, \ldots, a_n) := \mathrm{td}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) - \mathrm{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n).$$

Conjecture 1.3 ((SC), version 3)**.** *Let $a_1, \ldots, a_n$ be complex numbers. Then* $\delta(a_1, \ldots, a_n) \geqslant 0$.

The predimension function can be relativised. For a subset $A \subseteq \mathbb{C}$, we can define

$$\delta(a_1, \ldots, a_n/A) := \mathrm{td}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}/A, \exp(A)) - \mathrm{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n/A).$$

For example, we can take $A = \ker(\exp) = 2\pi i\mathbb{Z}$, and that gives a version of Schanuel's conjecture "over the kernel".

Conjecture 1.4 (SC over the kernel)**.** *Let $a_1, \ldots, a_n$ be complex numbers. Then $\delta(a_1, \ldots, a_n/\ker(\exp)) \geqslant 0$.*

There is a weaker version of (SC) which also ignores the kernel.

Conjecture 1.5 ((Weak SC))**.** *Let $a_1, \ldots, a_n$ be complex numbers and suppose that $\operatorname{td}_{\mathbb{Q}}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) < n$. Then there are integers $m_1, \ldots, m_n$, not all zero, such that $\prod_{i=1}^{n} e^{m_i a_i} = 1$.*

Proposition 1.6. *(SC) $\implies$ (SC over the kernel) $\implies$ (Weak SC).*

The converse implication (Weak SC) $\implies$ (SC over the kernel) is false. For example, if e and π were algebraically dependent, but Schanuel's conjecture had no other counterexamples, then (Weak SC) would be true but $\delta(1/\ker) = -1$.

The implication (SC over the kernel) $\implies$ (SC) is true, because $2\pi i$ is transcendental, and the kernel is a cyclic group. However the equivalent implication is not necessarily true for other exponential fields.

Geometric statements

In a different direction, we can use the fact that the transcendence degree of an m-tuple of complex numbers is smaller than n iff the tuple lies in an algebraic variety of dimension less than n.

Conjecture 1.7 ((SC), version 4)**.** *Let $a_1, \ldots, a_n$ be complex numbers, and suppose that the $2n$-tuple $(a_1, e^{a_1}, \ldots, a_n, e^{a_n})$ lies in an algebraic subvariety V of $\mathbb{C}^{2n}$ which is defined over $\mathbb{Q}$ and of dimension strictly less than n. Then there are integers $m_1, \ldots, m_n$, not all zero, such that $\sum_{i=1}^{n} m_i a_i = 0$.*

Continuing in this direction, we can find a more geometric statement. Write $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$ for the multiplicative group of the complex numbers, and $\mathbb{G}_{\mathrm{a}}(\mathbb{C})$ for its additive group. $\mathbb{G}_{\mathrm{a}}(\mathbb{C})$ is just $\mathbb{C}$, and is naturally identified with the universal covering space of $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$, and the exponential map is the covering map $\mathbb{G}_{\mathrm{a}}(\mathbb{C}) \xrightarrow{\exp} \mathbb{G}_{\mathrm{m}}(\mathbb{C})$.

$\mathbb{G}_{\mathrm{a}}(\mathbb{C})$ is also naturally identified with tangent space of $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$ at the identity. This is the Lie algebra of $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$, written $L\mathbb{G}_{\mathrm{m}}(\mathbb{C})$. The tangent bundle $T\mathbb{G}_{\mathrm{m}}(\mathbb{C})$ is naturally isomorphic to $L\mathbb{G}_{\mathrm{m}}(\mathbb{C}) \times \mathbb{G}_{\mathrm{m}}(\mathbb{C})$. Thus the graph of the exponential map is an analytic subvariety of the tangent bundle $\mathcal{G} \subseteq T\mathbb{G}_{\mathrm{m}}(\mathbb{C})$.

Conjecture 1.8 ((SC), version 5)**.** *Let V be an algebraic subvariety of $T\mathbb{G}_{\mathrm{m}}(\mathbb{C})^n$, defined over $\mathbb{Q}$, of dimension strictly less than n. Then the intersection $\mathcal{G} \cap V$ is contained in the union*

$$\bigcup \left\{ TH \mid H \text{ is a proper algebraic subgroup of } \mathbb{G}_{\mathrm{m}}(\mathbb{C})^n \right\}.$$

Power series version

For any ring R, the ring of formal power series over R is given by

$$R[\![t]\!] = \left\{ \sum_{n\in\mathbb{N}} a_n t^n \mid a_n \in R \right\}$$

with the usual term-by-term addition and multiplication. If f and g are two power series and g has no constant term, then there is a formal composite $f(g)$. The exponential function is represented by the formal power series $\sum_{n\in\mathbb{N}} \frac{t^n}{n!}$, and so the ring $R[\![t]\!]$ admits a partial exponential map defined on the subset $tR[\![t]\!]$, that is, on the principal ideal generated by t. In the case where R is an exponential ring, in particular for $\mathbb{C}_{\exp}$, the exponential map can be extended to the whole of $R[\![t]\!]$. For $a = \sum_{n\in\mathbb{N}} a_n t^n$, define $\exp(a) = \exp(a_0)\exp(a - a_0)$, where the first exp is that defined on R and the second is that defined on power series with no constant term. We can also consider power series in several variables $t_1, \ldots, t_r$. There are the usual derivations $\frac{\partial}{\partial t_j}$ for $j = 1, \ldots, r$. The *Jacobian matrix* of a tuple $f = (f_1, \ldots, f_n)$ is $\mathrm{Jac}(f) = \left(\frac{\partial f_i}{\partial t_j}\right)_{i,j}$. The rank of this matrix is used.

The analogue of Schanuel's conjecture for power series is as follows. The special case of this statement when $r = 0$ is just (SC). In theorem 2 of [Ax71], James Ax proved the converse implication.

Conjecture 1.9 ((SC), power series version)**.** *Let $f_1, \ldots, f_n \in \mathbb{C}[\![t_1, \ldots, t_r]\!]$. If $\mathrm{td}_{\mathbb{C}}(f_1, \exp(f_1), \ldots, f_n, \exp(f_n)) - \mathrm{rk}\,\mathrm{Jac}(f) < n$ then there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i f_i = 0$.*

We can also consider the ring of convergent power series $\mathbb{C}\{\{t\}\}$, the subring of $\mathbb{C}[\![t]\!]$ consisting of those power series with non-zero radius of convergence. $\mathbb{C}\{\{t\}\}$ is naturally a (total) exponential ring. The restriction of the above power series version of (SC) to convergent power series is intermediate between (SC) and the power series statement, hence it is equivalent to both.

Roy's version

Damien Roy [Roy01] has shown that (SC) is equivalent to the following statement, which is in the style of transcendental number theoretic statements saying that transcendence is equivalent to having a good rational approximation.

Conjecture 1.10. *Let $n \in \mathbb{N}$, and fix positive real numbers s_0, s_1, t_0, t_1, u such that $\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$ and*

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Then for all $x_1, \ldots, x_n \in \mathbb{C}$ *and all* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^\times$*, either*

- *the* x_i *are* $\mathbb{Q}$*-linearly dependent, or*
- $\mathrm{td}_{\mathbb{Q}}(\bar{x}, \bar{\alpha}) \geqslant n$*, or*
- *there is* $N_0 \in \mathbb{N}$ *such that for all* $N \in \mathbb{N}$ *greater than* N_0 *there is a nonzero polynomial* $P_N \in \mathbb{Z}[X_0, X_1]$ *with partial degrees at most* N^{t_0} *in* X_0 *and* N^{t_1} *in* X_1*, and all coefficients of modulus at most* e^N *such that for all* $k, m_1, \ldots, m_n \in \mathbb{N}$ *with* $k \leqslant N^{s_0}$ *and* $\max\{m_1, \ldots, m_n\} \leqslant N^{s_1}$ *we have*
$$\left| (\mathcal{D}^k P_N) \left(\sum_{j=1}^n m_j x_j, \prod_{j=1}^n \alpha_j^{m_j} \right) \right| \leqslant e^{-N^u}$$

where $\mathcal{D}$ *is the derivation* $\frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$.

2 Known cases

The following are known special cases of Schanuel's conjecture.

Hermite, 1873 e is transcendental. (Special case of (SC) with $n = 1, a_1 = 1$.)

Lindemann, 1882 π is transcendental. ($n = 1, a_1 = \pi i$.)

Lindemann, 1882 If a is algebraic then e^a is transcendental. ($n = 1$)

Lindemann-Weierstrass theorem, 1885 If $a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly independent algebraic numbers then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent. (All the a_i are algebraic.)

Nesterenko, 1996 π and e^π are algebraically independent. ($n = 2, a_1 = \pi, a_2 = i\pi$.)

However, some very simple cases are unknown. For example, it is not known if e and π are algebraically independent. ($n = 2, a_1 = 1, a_2 = \pi i$.)

There are other known theorems which give special cases, such as the Gelfond-Schneider theorem and Baker's theorem.

3 Uniform SC and Strong SC

Schanuel's conjecture was strengthened to a uniform version by Zilber in [Zil02].

Conjecture 3.1 ((USC), Uniform Schanuel Conjecture)**.** *For each algebraic subvariety $V \subseteq T\mathbb{G}_m(\mathbb{C})^n$ defined over $\mathbb{Q}$ and of dimension strictly less than n, there is a finite set $\mathcal{H}_V$ of proper algebraic subgroups of ${\mathbb{G}_m}^n$ such that for any $a_1, \ldots, a_n \in \mathbb{C}$, if $(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) \in V$ then there is $H \in \mathcal{H}_V$ such that $(a_1, \ldots, a_n) \in LH + 2\pi i\mathbb{Z}^n$, and either H has codimension at least 2, or $(a_1, \ldots, a_n) \in LH$.*

This conjecture can be written in an intersection version similar to Conjecture 1.8, but it is a little more messy.

It is easy to see from the statements that (USC) implies (SC). The converse is not known, but the diophantine conjecture CIT (see later) implies that any exponential field satisfying (SC) does indeed satisfy (USC). However, we can do without CIT in some cases. Write ($SC_{\mathbb{R}}$) for the conjecture that the real exponential field $\mathbb{R}_{\exp}$ satisfies (SC), that is that the statement of Schanuel's conjecture holds when the a_i are real numbers. Similarly, write ($USC_{\mathbb{R}}$) for the conjecture that $\mathbb{R}_{\exp}$ satisfies (USC).

Proposition 3.2 ([KZ06])**.** *($SC_{\mathbb{R}}$) $\iff$ ($USC_{\mathbb{R}}$). Moreover, if an exponential field (or partial exponential field) is o-minimal with analytic cell-decomposition and satisfies (SC) then it also satisfies (USC).*

The structure of $\mathbb{R}_{\exp}$ with restricted sine interprets the complex field with complex exponentiation defined on the strip $\mathbb{R} \times [-\pi i, \pi i]$. This structure is known to be o-minimal, with analytic cell-decomposition.

Proof. The proof of this for $\mathbb{R}_{\exp}$ is in [KZ06], but it only uses the fact that $\mathbb{R}_{\exp}$ is o-minimal and has analytic cell-decomposition, so the same proof gives the full statement. □

Another strengthening of (SC) is the following.

Conjecture 3.3 ((SSC), Strong Schanuel conjecture)**.** *The statement of (SC) holds for all ultrapowers of $\mathbb{C}_{\exp}$ (including $\mathbb{C}_{\exp}$ itself). Equivalently, (SC) is true of $\mathbb{C}_{\exp}$ and is part of its first order theory.*

Added in 2018: The conjecture (SSC) is false. See [KZ14] for a discussion of the issue.

4 Parametric SC

(SC) version 4 applies to subvarieties defined over $\mathbb{Q}$, but there is a statement dealing with subvarieties defined over any subfield of $\mathbb{C}$.

Conjecture 4.1 ((SC_{param}), Schanuel conjecture with parameters)**.** *Let $V \subseteq T\mathbb{G}_{\mathrm{m}}(\mathbb{C}))^n$ be any algebraic subvariety, of dimension strictly less than n. Then there is $l \in \mathbb{N}$ and $b_1, \ldots, b_l \in \mathbb{C}$ such that if $a_1, \ldots, a_n \in \mathbb{C}$ such that $(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) \in V$ then there are $m_1, \ldots, m_{n+l} \in \mathbb{Z}$, with $m_1, \ldots, m_n$ not all zero, such that $\sum_{i=1}^{n} m_i a_i + \sum_{i=1}^{l} m_{i+n} b_i = 0$.*

Furthermore, if V is a fibre $W(\bar{p})$ where $W \subseteq T\mathbb{G}_{\mathrm{m}}(\mathbb{C})^n \times P$ and P is an algebraic subvariety of $\mathbb{C}^k$ defined over $\mathbb{Q}$ then we may take $l = k(n+1)$.

Proposition 4.2 ([Zil02, Proposition 4])**.** *(SC) $\implies$ (SC_{param})*

The uniform version is simpler to state.

Conjecture 4.3 ((USC_{param}), Uniform Schanuel conjecture with parameters)**.**
Let $V \subseteq T\mathbb{G}_{\mathrm{m}}(\mathbb{C}))^n$ be any algebraic subvariety, of dimension strictly less than n. Then there is a finite collection $\mathcal{K}_V$ of cosets of the form $g \cdot TH$ where H is a proper algebraic subgroup of $\mathbb{G}_{\mathrm{m}}(\mathbb{C})^n$ such that $\mathcal{G} \cap V \subseteq \bigcup \mathcal{K}_V$.

5 Ax's theorem and some consequences

An important piece of work on Schanuel's conjecture was done by James Ax, in [Ax71], using differential fields. Let F be a field (of characteristic zero), Δ a set of derivations on F, and C the common field of constants, which is given by $C = \bigcap_{D \in \Delta} \{x \in F \mid Dx = 0\}$. Usually Δ will be a finite set or a finite-dimensional F-vector space of derivations. If $x = (x_1, \ldots, x_n)$ is a finite tuple of elements of F, the *Jacobian matrix* of x is defined to be the matrix $\mathrm{Jac}(x)$ with entries (Dx_i) for $i = 1, \ldots, n$ and $D \in \Delta$. The order of the columns will not matter, nor that the matrix is infinite if Δ is infinite, because we are only concerned with the rank of the matrix, $\mathrm{rk}\,\mathrm{Jac}(x)$, which is at most n. Write

$$\Gamma = \left\{ (x, y) \in T\mathbb{G}_{\mathrm{m}}(F) \;\middle|\; \frac{Dy}{y} = Dx \text{ for each } D \in \Delta \right\}$$

for the solution set to the exponential differential equation. In this language, Ax's main theorem is as follows.

Theorem 5.1 ((SC_D), Differential field Schanuel condition, Ax's theorem)**.** *If $x_1, y_1, \ldots, x_n, y_n \in \Gamma^n$ and $\mathrm{td}_C(x_1, y_1, \ldots, x_n, y_n) - \mathrm{rk}\,\mathrm{Jac}(x) < n$ then there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^{n} m_i x_i \in C$ and $\prod_{i=1}^{n} y_i^{m_i} \in C$.*

An equivalent statement is the power series version of (SC), restricted to power series with no constant term.

Theorem 5.2 ([Ax71, Corollary 1]). *Let C be a field of characteristic zero and let $f_1, \ldots, f_n \in C[\![t_1, \ldots, t_r]\!]$ be power series with no constant term. If* $\mathrm{td}_C(f_1, \exp(f_1), \ldots, f_n, \exp(f_n)) - \mathrm{rk}\,\mathrm{Jac}(f) < n$ *then there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i f_i = 0$.*

An immediate corollary translates this into a result about analytic functions.

Theorem 5.3 ([Ax71, Corollary 2]). *Let C be an algebraically closed field of characteristic zero which is complete with respect to a non-discrete absolute value. Let $f_1, \ldots, f_n$ be analytic functions in some polydisc about the origin 0 in C^r such that each $\exp(f_i)$ is defined and for which the $f_i - f_i(0)$ are $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{td}_C(f_1, \exp(f_1), \ldots, f_n, \exp(f_n)) - \mathrm{rk}\,\mathrm{Jac}(f) \geqslant n.$$

Robert Coleman strengthened theorem 5.2 in [Col80], in the special case that the power series are defined over $\bar{\mathbb{Q}}$. For a field F, write $F(\!(t)\!)$ for the field of Laurent series, that is, the field of fractions of $F[\![t]\!]$. Let $\mathcal{O}$ be the ring of algebraic integers (a subring of $\bar{\mathbb{Q}}$) and let k be its field of fractions.

Theorem 5.4. *Let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[\![t_1, \ldots, t_r]\!]$ be power series with no constant term, and suppose they are $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{td}_{k(\!(t_1, \ldots, t_r)\!)}(f_1, \exp(f_1), \ldots, f_n, \exp(f_n)) \geqslant n.$$

The uniform version of Ax's theorem is also a theorem.

Theorem 5.5 ((USC$_D$), [Kir06]). *For each parametric family $(V_c)_{c \in P(C)}$ of subvarieties of $T{\mathbb{G}_\mathrm{m}}^n S$, with V_c defined over $\mathbb{Q}(c)$, there is a finite set $\mathcal{H}_V$ of proper algebraic subgroups of ${\mathbb{G}_\mathrm{m}}^n$ such that for each $c \in P(C)$ and each $(x, y) \in \Gamma^n \cap V_c$, if $\dim V_c - \mathrm{rk}\,\mathrm{Jac}(x, y) < n$, then there is $\gamma \in TS(C)$ and $H \in \mathcal{H}_V$ such that (x, y) lies in the coset $\gamma \cdot TH$.*

A corollary deals with complex analytic varieties.

Theorem 5.6 ([Kir06, Theorem 8.1]). *Let P be an algebraic variety and $(V_p)_{p \in P(\mathbb{C})}$ be a parametric family of algebraic subvarieties of $T\mathbb{G}_\mathrm{m}(\mathbb{C})^n$. There is a finite collection $\mathcal{H}_V$ of proper algebraic subgroups of ${\mathbb{G}_\mathrm{m}}^n$ with the following property:*

If $p \in P$ and W is a connected component of the analytic variety $\mathcal{G} \cap V_p$ with analytic dimension $\dim W$ satisfying $\dim W > \dim V_p - n$, then there is $H \in \mathcal{H}_V$ and $g \in T\mathbb{G}_\mathrm{m}(\mathbb{C})^n$ such that W is contained in the coset $g \cdot TH$.

6 Two-sorted exponentiation

Consider the two-sorted structure of the exponential map with the domain and codomain as separate copies of $\mathbb{C}$, with the full field structure on each. The difference with the one-sorted case is that we do not have a chosen isomorphism between the two sorts. In this setting, we cannot ask about algebraic relations between elements a_i of the domain and their images e^{a_i} in the codomain. Thus the relevant version of Schanuel's conjecture in this setting is weaker. [Zil00]

Conjecture 6.1 ((2-sorted SC)). *Let $a_1, \ldots, a_n$ be complex numbers. Then*

$$\mathrm{td}_{\mathbb{Q}}(a_1, \ldots, a_n) + \mathrm{td}_{\mathbb{Q}}(e^{a_1}, \ldots, e^{a_n}) - \mathrm{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n) \geqslant 0.$$

Proposition 6.2. *(SC) $\implies$ (2-sorted SC)*

Proof. $\mathrm{td}_{\mathbb{Q}}(a_1, \ldots, a_n) + \mathrm{td}_{\mathbb{Q}}(e^{a_1}, \ldots, e^{a_n}) \geqslant \mathrm{td}_{\mathbb{Q}}(a_1, e^{a_1}, \ldots, a_n, e^{a_n})$. □

7 Raising to powers

The exponential function is used to define the (multivalued) functions of raising to a power. For $a, b, \rho \in \mathbb{C}$, write $b = a^\rho$ to mean that there is $\alpha \in \mathbb{C}$ such that $e^\alpha = a$ and $e^{\rho\alpha} = b$. Schanuel's conjecture (for exponentiation) implies statements about these "raising to powers" functions. We say that $x_1, \ldots, x_n$ are *multiplicatively dependent* iff there are $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^n x_i^{m_i} = 1$. Otherwise they are *multiplicatively independent*.

Algebraic powers

Conjecture 7.1 ((SCP_{alg}), Schanuel Conjecture for raising to an algebraic power). *Let $\rho \in \bar{\mathbb{Q}}$, and $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}$ such that $b_i = a_i^\rho$. If we have $\mathrm{td}_{\mathbb{Q}}(a_1, b_1, \ldots, a_n, b_n) < n$ then there are $m_1, \ldots, m_{2n} \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^n a_i^{m_i} b_i^{m_{n+i}} = 1$.*

Equivalently, if the a_i and b_i are multiplicatively independent then we have $\mathrm{td}_{\mathbb{Q}}(a_1, b_1, \ldots, a_n, b_n) \geqslant n$.

If ρ is rational then this statement is uninteresting.

Proposition 7.2. *(Weak SC) $\implies$ (SCP_{alg})*

Proof. For each i let α_i be such that $e^{\alpha_i} = a_i$ and $e^{\rho\alpha_i} = b_i$. If we have $\mathrm{td}_{\mathbb{Q}}(a_1, b_1, \ldots, a_n, b_n) < n$ then $\mathrm{td}_{\mathbb{Q}}(\alpha_1, a_1, \ldots, \alpha_n, a_n, \rho\alpha_1, b_1, \rho\alpha_n, b_n) < 2n$, so by (Weak SC) there are $m_1, \ldots, m_{2n} \in \mathbb{Z}$, not all zero, such that we have $\prod_{i=1}^n a_i^{m_i} b_i^{m_{n+i}} = 1$, as required. □

The $n = 1$ case is known.

Theorem 7.3 (Gelfond-Schneider theorem, 1934)**.** *If a, ρ are algebraic, $b = a^\rho$, $a \neq 0, 1$, and ρ is irrational, then b is transcendental. (The special case of (SCP$_{alg}$) with $n = 1$.)*

Generic powers

See the paper [BKW10] for a proof of the conjecture discussed in this section, in a much improved and generalised form.

There is no "correct" simple form of a conjecture for raising to any power. For example, let $r = \log(3)/\log(2)$ in $\mathbb{R}$. Then $2^r = 3$, but $\mathrm{td}_{\mathbb{Q}}(2, 3) = 0$. However, there is a simple form of the statement when the power is generic, that is, not exponentially algebraic.

Definition 7.4 ([Mac96])**.** Let $K_{\exp} = \langle K; +, \cdot, \exp\rangle$ be an exponential field, and let $B \subseteq K$. An element α of K is said to be *exponentially algebraic* over B in $K_{\exp}$ iff there are $n, m \in \mathbb{N}$, tuples $a = (a_1, \ldots, a_n) \in K^n$, $b = (b_1, \ldots, b_m) \in B^m$, and $f_1, \ldots, f_n$ in the ring of exponential polynomials $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_m]^E$ such that:

i) $a_1 = \alpha$,

ii) For each $i = 1, \ldots, n$, $f_i(a, b) = 0$, and

iii) $\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} (a, b) \neq 0$

When $B = \emptyset$ we say that α is *exponentially algebraic* in $K_{\exp}$.

Warning: The notion of exponential algebraicity depends on the exponential field. For example, assuming Schanuel's conjecture there are real numbers which are exponentially algebraic in $\mathbb{C}_{\exp}$ but are not exponentially algebraic in $\mathbb{R}_{\exp}$. Schanuel's conjecture even implies that π is such a number.

Conjecture 7.5 ((SCP$_{gen}$), Schanuel Conjecture for raising to a generic power)**.** *Let $\rho \in \mathbb{C}$, not exponentially algebraic in $\mathbb{C}_{\exp}$, and let $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}$ such that $b_i = a_i^\rho$. If the a_i and b_i are multiplicatively independent then $\mathrm{td}_{\mathbb{Q}}(a_1, b_1, \ldots, a_n, b_n) \geqslant n$.*

Proposition 7.6. *(SC) $\implies$ (SCP$_{gen}$)*

Sketch proof. Let α_i be such that $e^{\alpha_i} = a_i$. Suppose $\mathrm{td}_{\mathbb{Q}}(a, b) < n$. Then $\mathrm{td}_{\mathbb{Q}}(\alpha, \rho, a, b) < 2n + 1$. So $\delta(\alpha, \rho\alpha) < 2n + 1 - \mathrm{ldim}_{\mathbb{Q}}(\alpha, \rho\alpha)$.

But $\delta(\alpha, \rho\alpha) \geqslant 1$, using (SC) and the fact that ρ is not exponentially algebraic, so $\mathrm{ldim}_{\mathbb{Q}}(\alpha, \rho\alpha) < 2n$. Thus a, b are multiplicatively dependent. □

Alex Wilkie proved two results about raising to generic real powers.

Theorem 7.7 ([Wil03b]). *Let r be a real number not definable in $\mathbb{R}_{\exp}$ (equivalently, r is not exponentially algebraic in $\mathbb{R}_{\exp}$), and let $a_1, \ldots, a_n$ be positive real numbers. If the set $\{a_1, a_1^r, \ldots, a_n, a_n^r\}$ is multiplicatively independent then $\mathrm{td}_{\mathbb{Q}}(a_1, a_1^r, \ldots, a_n, a_n^r) \geqslant n$.*

Theorem 7.8 ([Wil03a]). *Let $\mathcal{R}$ be the structure $\langle \mathbb{R}; +, \cdot, \exp, \sin\restriction_{[-\pi,\pi]} \rangle$, and let r be a real number not definable in $\mathcal{R}$. (Equivalently, r is not exponentially algebraic in $\mathbb{C}_{\exp}$.) If $a_1, \ldots, a_n$ are multiplicatively independent real numbers then $\mathrm{td}_{\mathbb{Q}}(a_1^{1+ir}, \ldots, a_n^{1+ir}) \geqslant n/2$.*

These statements were obtained using Ax's theorem $(\mathrm{SC})_D$. Uniform versions can be obtained by using $(\mathrm{USC})_D$, and this is done in the second case in [Wil03a].

Non-standard integer powers

See chapter 6 of Martin Bays' thesis [Bay09] for a better account of non-standard integer powers.

There is a notion of raising to a non-standard integer power. In $\mathbb{C}_{\exp}$, a number z lies in $\mathbb{Z}$ iff it satisfies the formula

$$\varphi(z) \equiv \forall y[e^y = 1 \to e^{zy} = 1].$$

In any exponential field, the map $x \mapsto x^\rho$ is a single-valued function precisely when $\varphi(\rho)$ holds.

A *non-standard integer* in an exponential field $K_{\exp}$ with a non-trivial kernel is an element $\rho \in K_{\exp}$ such that $K_{\exp} \models \varphi(\rho)$, but which is not in $\mathbb{Z}$ (the standard integers). There are elementary extensions of $\mathbb{C}_{\exp}$ (for example, ultrapowers), in where there are non-standard integers.

For an exponential field $K_{\exp}$ with a non-standard integer ρ, we isolate the following property.

(SCP_{nsi}), SC for raising to a non-standard integer power .

If $a_1, \ldots, a_n \in K_{\exp}$ are such that the a_i and a_i^ρ are multiplicatively independent then $\mathrm{td}_{\mathbb{Q}}(a_1, a_1^\rho, \ldots, a_n, a_n^\rho) \geqslant n$.

Conjecture 7.9. *(SCP_{nsi}) holds for all non-standard integers in all ultrapowers of $\mathbb{C}_{\exp}$.*

Note that if k is a nonzero kernel element and ρ is a non-standard integer then $\exp(k) = 1$ shows that k is exponentially algebraic, and then $\exp(\rho k) = 1$ shows that ρ is exponentially algebraic. So (SCP$_{nsi}$) and (SCP$_{gen}$) are independent statements.

A field of powers

It is natural to consider not just a single (multivalued) power function, but a field of powers. This was done in Zilber's paper [Zil03]. This setup can be captured by another two-sorted model, $\mathbb{C} \xrightarrow{\exp} \mathbb{C}$, in this case where the domain sort has the structure of a K-vector space for some field K, not the full field structure. We take K to be a subfield of $\mathbb{C}$ of finite transcendence degree d, and the domain to have the natural K-vector space structure. The Schanuel conjecture for this setting is as follows.

Conjecture 7.10 ((SCP$_K$), SC for raising to powers from the field K)**.** *Let $a_1, \ldots, a_n$ be complex numbers. Then*

$$\text{K-ldim}(a_1, \ldots, a_n) + \text{td}_{\mathbb{Q}}(e^{a_1}, \ldots, e^{a_n}) - \text{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n) \geqslant -d.$$

Proposition 7.11. *(2-sorted SC) $\implies$ (SCP$_K$)*

Proof.

$$\begin{aligned} \text{K-ldim}(a_1, \ldots, a_n) + d &\geqslant \text{td}_K(a_1, \ldots, a_n) + d \\ &\geqslant \text{td}_K(a_1, \ldots, a_n) + \text{td}_{\mathbb{Q}}(K) \\ &\geqslant \text{td}_{\mathbb{Q}}(a_1, \ldots, a_n) \end{aligned}$$

□

Proposition 7.12. *(SCP$_{\bar{\mathbb{Q}}}$) $\implies$ (SCP$_{alg}$)*

Proof. Let $\rho \in \bar{\mathbb{Q}}$, and $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}$ such that $b_i = a_i^{\rho}$, and suppose the a_i and b_i are multiplicatively independent. For each i, choose $\alpha_i \in \mathbb{C}$ such that $\exp(\alpha_i) = a_i$. Then the α_i and $\rho\alpha_i$ are $\mathbb{Q}$-linearly independent. Since $K = \mathbb{Q}(\rho)$, we have $d = 0$. Thus from (SCP$_{\bar{\mathbb{Q}}}$) we get

$$\text{K-ldim}(\bar{\alpha}, \rho\bar{\alpha}) + \text{td}_{\mathbb{Q}}(\bar{a}, \bar{b}) - \text{ldim}_{\mathbb{Q}}(\bar{\alpha}, \rho\bar{\alpha}) \geqslant 0.$$

The α_i and $\rho\alpha_i$ are $\mathbb{Q}$-linearly independent, and $\text{K-ldim}(\bar{\alpha}, \rho\bar{\alpha}) \leqslant n$, so this becomes

$$\text{td}_{\mathbb{Q}}(\bar{a}, \bar{b}) - 2n \geqslant -n$$

which gives the result. □

However, if ρ is a non-algebraic power then, taking $K = \mathbb{Q}(\rho)$ we have $d = 1$, and so (SCP$_{\mathbb{Q}(\rho)}$) does not imply (SCP$_{gen}$), nor (SCP$_{nsi}$).

8 The Conjecture on Intersections of Tori with algebraic varieties

Added in 2018: The CIT is now commonly known as the multiplicative case of the Zilber-Pink conjecture.

Boris Zilber gave in [Zil02] a conjecture on intersections of tori with algebraic varieties (CIT) which is connected to the uniformity in Schanuel conditions. It has a purely algebraic (Diophantine) statement, as opposed to the statements above which involve some sort of analytic exponential map. Surprisingly, it can also be seen as a Schanuel condition itself.

As before, we identify algebraic varieties with their complex points.

Definition 8.1. If U is a smooth, irreducible algebraic variety, V, W are irreducible subvarieties of U and X is an irreducible component of $V \cap W$ then X is said to be a *typical component* of the intersection iff $\dim X = \dim V + \dim W - \dim U$ and it is said to be an *atypical component* iff $\dim X > \dim V + \dim W - \dim U$.

Since U is smooth, and we are considering the points in an algebraically closed field, it is impossible to have $\dim X < \dim V + \dim W - \dim U$.

Conjecture 8.2 ((CIT))**.** *For any algebraic subvariety V of $G = \mathbb{G}_{\mathrm{m}}{}^n(\mathbb{C})$, defined over $\bar{\mathbb{Q}}$, there is a finite collection $\mathcal{H}_V$ of proper algebraic subgroups of G such that for any algebraic subgroup T of G, if X is an atypical component of $V \cap T$ in G then there is $H \in \mathcal{H}_V$ such that $X \subseteq H$.*

There is a version for parametric families of subvarieties as well.

Conjecture 8.3 ((CIT$_{param}$), CIT with parameters)**.** *Let P be a complex algebraic variety and let $(V_p)_{p \in P}$ be a parametric family of subvarieties of $G = \mathbb{G}_{\mathrm{m}}{}^n(\mathbb{C})$, the family defined over $\bar{\mathbb{Q}}$.*

Then there is a finite collection $\mathcal{H}_V$ of proper algebraic subgroups of G, a natural number $t = t_V$ and functions $c_i : P \times G \to G$ for $i = 1, \ldots, t$ such that for any coset $T \cdot g$ of any algebraic subgroup of G, if X is an atypical component of $V_p \cap T \cdot g$ in G then there is $H \in \mathcal{H}_V$ and $i \in \{1, \ldots, t\}$ such that $X \subseteq H \cdot c_i(p, g)$.

Proposition 8.4 ([Zil02, Theorem 1, Corollary 1])**.**
(CIT) $\implies$ (CIT$_{param}$)

It is not clear that the converse implication holds.

A weak version of the CIT (with parameters), not giving the natural number t, is a theorem. Proofs are given in [Zil02] and [Kir06].

Theorem 8.5 (Weak CIT). *Let P be a complex algebraic variety and let $(V_p)_{p\in P}$ be a parametric family of subvarieties of $G = \mathbb{G}_{\mathrm{m}}{}^n(\mathbb{C})$, the family defined over $\bar{\mathbb{Q}}$.*

Then there is a finite family $\mathcal{H}_V$ of proper algebraic subgroups of G such that, for any coset $\kappa = a \oplus T$ of any algebraic subgroup T of G and any $p \in P$, if X is an atypical component of the intersection, then there is $H \in \mathcal{H}_V$ and $c \in G$ such that $X \subseteq H \cdot c$.

Proposition 8.6 ([Zil02, Proposition 5]). *(SC)* & *(CIT)* $\implies$ *(USC)*

9 Statements for other algebraic groups

Schanuel's conjecture is a statement about the exponential map of $\mathbb{G}_{\mathrm{m}}$. Since we have several variables, it is more properly seen as a statement about the exponential maps of the *algebraic tori*, the algebraic groups of the form $\mathbb{G}_{\mathrm{m}}{}^n$. We can replace the torus by any (complex) commutative algebraic group, and still have an exponential map which is a group homomorphism.

Let S be a complex commutative algebraic group, n its dimension, let $LS \cong \mathbb{G}_{\mathrm{a}}{}^n$ be the tangent space at the identity (the Lie algebra) and $TS = LS \times S$ the tangent bundle. The exponential map $\exp_S$ is a group homomorphism $LS \xrightarrow{\exp_S} S$, and so its graph $\mathcal{G}_S$ is an analytic subgroup of TS.

In characteristic zero, there is a fairly simple description of commutative algebraic groups. Any connected such group S has a Chevalley decomposition, a short exact sequence

$$0 \to \mathbb{G}_{\mathrm{a}}{}^r \times \mathbb{G}_{\mathrm{m}}{}^l \hookrightarrow S \twoheadrightarrow A \to 0$$

where $r, l \in \mathbb{N}$ and A is an abelian variety.

- Vector groups: $l = 0$, $A = 0$
- Tori: $r = 0$, $A = 0$
- Elliptic curves: $r = 0$, $l = 0$, $\dim A = 1$
- Abelian varieties: $r = 0$, $l = 0$
- Semiabelian varieties: $r = 0$
- Groups with no vector quotient (nvq-groups): the extension by $\mathbb{G}_{\mathrm{a}}{}^r$ does not split. In this case, $r \leqslant \dim A$.

There are inclusions
Elliptic curves $\subseteq$ Abelian varieties $\subseteq$ Semiabelian varieties $\subseteq$ nvq-groups and also Tori $\subseteq$ Semiabelian varieties. However, the intersection between Tori and Abelian varieties is only the trivial group, and likewise the intersection between Vector groups and nvq-groups.

If S is a vector group, $LS = S$ and the exponential map is the identity map on S. Thus there is no interesting statement for vector groups.

The group S may not be defined over $\mathbb{Q}$, and the arithmetic of such groups is more complicated than that of $\mathbb{G}_{\mathrm{m}}$ involving periods, quasiperiods and so on, so formulating a "correct" equivalent of Schanuel's conjecture is much more difficult.

Bertolin [Ber02] gives a version where S is a product of a torus and elliptic curves, and shows that it follows from a conjecture on the periods of 1-motives by Grothendieck and André.

The parametric versions are easier to adapt. We assume that S has no vector quotients.

Conjecture 9.1 (($\mathrm{SC}^{nvq}_{param}$)). *Let $V \subseteq TS$ be an algebraic subvariety with $\dim V < \dim S$. Then there is a finitely generated subfield k of $\mathbb{C}$ such that if $(a, \exp_S(a)) \in V$ then there is a proper algebraic subgroup H of S and $c \in LS(k)$ such that $a \in c \cdot LH$.*

A statement of the full conjecture would at least involve specifying the field k in the case where V is defined over $\mathbb{Q}$.

Conjecture 9.2 (($\mathrm{USC}^{nvq}_{param}$)). *Let $V \subseteq TS$ be an algebraic subvariety with $\dim V < \dim S$. There is a finite collection $\mathcal{K}_V$ of cosets of the form $c \cdot TH$ where H is a proper algebraic subgroup of S such that $\mathcal{G}_S \cap V \subseteq \bigcup \mathcal{K}_V$.*

The analogues of (SC_D) and (USC_D) hold here too. We give the stronger statement.

As before, let F be a field of characteristic zero, Δ a set of derivations on F, and C the intersection of their constant subfields. For each $D \in \Delta$, let lD_S be the logarithmic derivative of S with respect to D, and similarly lD_{LS}. Let

$$\Gamma_{S,\Delta} = \{(x, y) \in (LS \times S)(F) \mid \mathrm{lD}_S(y) = \mathrm{lD}_{LS}(x) \text{ for all } D \in \Delta\}.$$

Let S be an nvq-group defined over C.

Theorem 9.3 ((USC^{nvq}_D)). *For each parametric family $(V_c)_{c\in P(C)}$ of subvarieties of TS, with V_c defined over $\mathbb{Q}(c)$, there is a finite set $\mathcal{H}_V$ of proper algebraic subgroups of S such that for each $c \in P(C)$ and each $(x, y) \in \Gamma^n \cap V_c$, if $\dim V_c - \mathrm{rk}\,\mathrm{Jac}(x, y) < \dim S$, then there is $\gamma \in TS(C)$ and $H \in \mathcal{H}_V$ such that (x, y) lies in the coset $\gamma \cdot TH$.*

When S is semiabelian, this is proved in [Kir09]. The proof there also adapts to the nvq-group case, which was first proved by Daniel Bertrand.

As in the torus case, there is the following analytic corollary. As with CIT, it is a statement about atypical intersections.

Theorem 9.4 ([Kir06, Theorem 8.1]). *Let P be an algebraic variety and $(V_c)_{c\in P(\mathbb{C})}$ be a parametric family of algebraic subvarieties of TS. There is a finite collection $\mathcal{H}_V$ of proper algebraic subgroups of S with the following property:*

If $c \in P(\mathbb{C})$ and W is a connected component of the analytic variety $\mathcal{G}_S \cap V_c$ with analytic dimension $\dim W$ satisfying $\dim W + \dim S - \dim V_c = t > 0$, then there is $H \in \mathcal{H}_V$ of codimension at least t and $g \in TS(\mathbb{C})$ such that W is contained in the coset $g \cdot TH$.

James Ax proved a version of (SC_D) in the context of arbitrary complex algebraic groups in [Ax72]. This theorem and the usual exponential (SC_D) are currently known as Ax-Schanuel theorems.

There are many possible analogues of the "raising to powers" statements. Some differential fields versions have been proved. We do not list them here.

The CIT naturally extends to semiabelian varieties, and was given in this context in [Zil02]. The weak version, below, is a theorem proved in [Kir09].

Theorem 9.5 (Weak semiabelian CIT). *Let S be a complex semiabelian variety, P be a complex algebraic variety and let $(V_p)_{p\in P}$ be a parametric family of subvarieties of S, the family defined over $\bar{\mathbb{Q}}$.*

Then there is a finite family $\mathcal{H}_V$ of proper algebraic subgroups of S such that, for any coset $\kappa = a \oplus T$ of any algebraic subgroup T of S and any $p \in P$, if X is an atypical component of the intersection, then there is $H \in \mathcal{H}_V$ and $c \in S$ such that $X \subseteq c \cdot H$.

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